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## Abstract

We consider a large collection of particles obeying $A_r$ statistics. The system behaves like a quantum droplet characterized by a constant Husimi distribution. We show that the excitations of this system live on the boundary of the droplet and they are described by an effective chiral boson action generalizing the Wess–Zumino–Witten theory in two dimension. Our analysis is based on the Fock–Bargmann analytical representations associated with $A_r$ statistics. The quantization of the theory describing the dynamics on the edge is achieved. As a by product, we prove that the edge excitations are given by a tensorial product of $r$ abelian bosonic fields.

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## 1. Introduction

Fifty years ago, a first extension of Bose and Fermi statistics was achieved by Green [1]. This extension was the basic underlying mathematical background to investigate the implications of the generalizations of the familiar bosonic and fermionic statistics [2–6]. In the last two decades a renewal interest has been devoted to generalized quantum statistics due to their possible relevance in some issues like for instance fractional quantum Hall effect [7, 8], anyon superconductivity [9] as well as black hole statistics [10]. Many variants of quantum statistics were proposed in the literature. One may quote the anyonic statistics [4] interpolating between fermionic and bosonic ones in two dimension space, quonic statistics [11] developed in the context of $q$-deformed algebras, $k$-fermionic statistics [12] defined as a $q$-deformed version of ordinary bosons when the deformation parameter is such that $q^k = 1$ and Haldane fractional statistics [13] to explain the origin of the fractional quantization of Hall conductivity.

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In the Green prescription [1], the generalized quantum statistics are characterized by certain triple relations which replace the commutation and anti-commutation rules for bosons and fermions. As a by product, the fermions and bosons are promoted to para-fermions and para-bosons of order $r$ respectively where the creation and annihilation operators satisfy

$$\left[\left[f_i^+, f_j^-\right], f_k^-\right] = -2\delta_{ik}f_j^-, \quad \left[\left[f_i^+, f_j^+\right], f_k^-\right] = -2\delta_{ik}f_j^+ + 2\delta_{jk}f_i^+, \quad \left[\left[f_i^-, f_j^-\right], f_k^-\right] = 0,$$
(1)

$$\left[\left\{b_i^+, b_j^-\right\}, b_k^-\right] = -2\delta_{ik}b_j^-, \quad \left[\left\{b_i^+, b_j^+\right\}, b_k^-\right] = -2\delta_{ik}b_j^+ - 2\delta_{jk}b_i^+, \quad \left[\left\{b_i^-, b_j^-\right\}, b_k^-\right] = 0,$$
(2)

with $i, j, k = 1, 2, \ldots, r$. From an algebraic point of view, the Grassmann algebra in the fermionic case is replaced by the para-fermionic one (1) which is related to the orthogonal Lie algebra $so(2r + 1) = B_r$ [14]. On the other side, the Weyl–Heisenberg algebra is extended to para-bosonic one (2) which is connected to the orthosymplectic superalgebra $osp(1/2r) = B(0, r)$ [15]. This indicates the deep link between Green statistics and the classical Lie and super Lie algebras. In this vein, very recently, on the basis of Palev works [6, 16], a classification of generalized quantum statistics was derived for the classical Lie algebras $A_r$, $B_r$, $C_r$ and $D_r$ [17].

This paper concerns the generalized $A_r$ statistics. This generalization incorporates two kinds of statistics. The first one deals with statistics satisfying a generalized exclusion Pauli principle and coincides with those derived by Palev [6–16]. This class will be termed here fermionic $A_r$ statistics. The second class is of bosonic kind. The particles can be accommodated in a given quantum state without any restriction. The generalization of $A_r$ statistics was derived by one of the authors [18–20]. However, for completeness and in order to fix our notations we shall, in the following section, review the definition of the generalized $A_r$ by means of the so-called Jacobson generators [21]. We use the trilinear relations defining the generalized quantum $A_r$ statistics to construct the corresponding Fock space. We give the actions of the corresponding creation and annihilation operators. The spectrum of the Hamiltonian, describing free particles obeying the generalized $A_r$ statistics, is determined. The analytical Bargmann representations corresponding to Fock space of bosonic as well as fermionic $A_r$ statistics are also presented. We give the differential realization of Jacobson operators. This analytical realization provides us with many advantages in discussing the semiclassical behavior of $A_r$ quantum systems. Indeed, in the Bargmann space viewed as the phase space of $A_r$ quantum systems, we study the excitations around a given droplet defined by a constant density (constant Husimi distribution). We show in this paper that the excitations live on the boundary of the droplet and are described by an effective action generalizing Wess–Zumino–Witten one, which describes chiral bosons in the two-dimensional spacetime [22]. To perform the semiclassical analysis of $A_r$ quantum systems and the derivation of the effective theory of the edge excitations, some tools are needed like the star product in the Bargmann space. The necessary material to do this is presented. The strategy that we adopt is closer to one followed by Das *et al* [23] and Sakita [24] for non-relativistic fermions localized around a ground many-body state and forming a droplet. The droplet is specified by a diagonal density matrix $\rho_0$. In the presence of an excitation potential, the fundamental state can be characterized by a unitary transformation of $\rho_0$, namely $\rho_0 \longrightarrow U\rho_0 U^\dagger$ where $U$ is a collective variable describing the possible excitations around the droplet.

The paper is organized as follows. In section 2, a brief review of generalized $A_r$ statistics, the associated Hamiltonian and the corresponding Fock space is given. Section 2 also deals with the Bargmann realization of the Jacobson generators (creation and annihilation operators of $A_r$ quantum systems). This provides us with a useful way to analyze semi-classically the

system under consideration. In this sense, we show in section 3 that a large collection of particles obeying $A_r$ statistics behaves like a droplet in the Bargmann space. In this semi-classical description, the operators are replaced by functions and the commutators become Moyal brackets. In section 4, we show that the excitations of the system live on the boundary of the droplet and they are described by a chiral boson theory. The advantages of such formulation lie on the fact that the dynamics of the obtained bosonic theory encodes the excitations of the $A_r$ statistical system. It follows that our formulation provides a first convenient step to study large collective states of $A_r$ statistics. Section 5 comprises concluding remarks.

## 2. The generalized $A_r$ statistics

In this section, we introduce the definitions of the Jacobson operators and the generalized $A_r$ statistics viewed as Lie triple system. We review the construction of the corresponding Fock space and we give the Hamiltonian describing a quantum system obeying generalized $A_r$ statistics [18] (see also [19, 20]).

### 2.1. Jacobson generators

First, let us introduce the notion of Lie triple system. Let $V$ be a vector space over a field $F$ which is assumed to be either real or complex. The vector space $V$ equipped vector with a trilinear mapping

$$[x, y, z] : V \otimes V \otimes V \longrightarrow V$$

is called Lie triple system if the following identities are satisfied:

$$[x, x, x] = 0,$$
$$[x, y, z] + [y, z, x] + [z, x, y] = 0,$$
$$[x, y, [u, v, w]] = [[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]].$$

According to this definition, we will introduce the generalized $A_r$ statistics as Lie triple system. In this respect, the algebra $\mathcal{G}$ defined by the generators $a_i^+$ and $a_i^-$ ($i = 1, 2, \ldots, r$) mutually commuting $\left([a_i^-, a_j^-] = [a_i^+, a_j^+] = 0\right)$ and satisfying the triple relation

$$[[a_i^+, a_j^-], a_k^+] = -s\delta_{jk}a_i^+ - s\delta_{ij}a_k^+ \tag{3}$$
$$[[a_i^+, a_j^-], a_k^-] = s\delta_{ik}a_j^- + s\delta_{ij}a_k^-, \tag{4}$$

where $s \in \{1, -1\}$, is closed under the ternary operation

$$[x, y, z] = [[x, y], z]$$

and defines a Lie triple system. The elements $a_i^\pm$ are termed Jacobson generators and will be identified later with creation and annihilation operators of a quantum system obeying generalized $A_r$ statistics. Note that for $s = -1$, the algebra $\mathcal{G}$ reduces to one defining the $A_r$ statistics discussed in [6]. As we will see in what follows, the sign of the parameter $s$ plays an importance in the representation of the algebra $\mathcal{G}$ and consequently, one can obtain different microscopic and macroscopic statistical properties of the quantum system under consideration. Finally, recall that these statistics are intrinsically related to simple Lie algebras of class $A$ like the para-fermion statistics which are related to class $B$ of simple Lie algebras.

### 2.2. The Hamiltonian

The Jacobson generators $a_i^{\pm}$ can be identified with creation and annihilation operators of a quantum gas obeying the generalized $A_r$ statistics. This requires a consistency with the Heisenberg equation

$$\left[H, a_i^{\pm}\right] = \pm e_i a_i^{\pm} \tag{5}$$

where $H$ is the Hamiltonian of the system and the quantities $e_i$ are the energies of the modes $i = 1, 2, \ldots, r$. One can verify that if $|E\rangle$ is an eigenstate with energy $E$, $a_i^{\pm}|E\rangle$ are eigenvectors of $H$ with energies $E \pm e_i$. In this respect, the operators $a_i^{\pm}$ can be interpreted as those creating or annihilating particles. To solve the consistency equation (5), we write the Hamiltonian $H$ as

$$H = e_0 \mathbf{1} + h = e_0 \mathbf{1} + \sum_{i=1}^{r} e_i h_i, \tag{6}$$

where $e_0$ is an arbitrary real constant and $\mathbf{1}$ is an operator commuting with all the elements of the algebra $\mathcal{G}$ (it can be viewed as the second-order Casimir operator). Using the structure relations of the algebra $\mathcal{G}$, the solution of the Heisenberg condition (5) is given by

$$h_i = \frac{s}{r+1}\left[(r+1)\left[a_i^-, a_i^+\right] - \sum_{j=1}^{r}\left[a_j^-, a_j^+\right]\right] + c \tag{7}$$

where the constant $c$ will be defined later such that the ground state (vacuum) of the Hamiltonian $H$ gives the energy $e_0$. The Hamiltonian $H$ seems to be a simple sum of 'free' (non-interacting) Hamiltonians $h_i$. However, it is important to note that in the quantum system under consideration, the statistical interactions occur and are encoded in the triple commutation relations (3) and (4).

### 2.3. Fock representations

A Hilbertian representation of the algebra $\mathcal{G}$ can be simply derived using the relation structures (3), (4) defining $A_r$ statistics. Here, we give the main results. For more details see [18, 20]. Since, the algebra $\mathcal{G}$ is spanned by $r$ pairs of Jacobson generators, it is natural to assume that the Fock space $\mathcal{F}$ is given by

$$\mathcal{F} = \oplus_{n=0}^{\infty} \mathcal{H}^n, \tag{8}$$

where

$$\mathcal{H}^n \equiv \left\{|n_1, n_2, \ldots, n_r\rangle, n_i \in \mathbf{N}, \sum_{i=1}^{r} n_i = n > 0\right\}$$

and $\mathcal{H}^0 \equiv \mathbf{C}$. The action of $a_i^{\pm}$, on $\mathcal{F}$, are defined by

$$a_i^{\pm}|n_1, \ldots, n_i, \ldots, n_r\rangle = \sqrt{F_i(n_1, \ldots, n_i \pm 1, \ldots, n_r)}|n_1, \ldots, n_i \pm 1, \ldots, n_r\rangle \tag{9}$$

where the functions $F_i$ are called the structure functions. Using the triple structure relations of $A_r$ statistics, one obtains [18, 20] the following expressions:

$$F_i(n_1, \ldots, n_i, \ldots, n_r) = \tfrac{1}{2}n_i(2k - (1+s) + 2s(n_1 + n_2 + \cdots + n_r)), \tag{10}$$

in terms of the quantum numbers $n_1, n_2, \ldots, n_r$. In equation (10), the real parameter labeling the obtained representation satisfies the condition $2k - 1 > s$. The dimension of the irreducible representation space $\mathcal{F}$ is determined by the condition:

$$k - \frac{1+s}{2} + s(n_1 + n_2 + \cdots + n_r) > 0. \tag{11}$$

It depends on the sign of the parameter $s$. It is clear that for $s = 1$, the Fock space $\mathcal{F}$ is infinite dimensional. However, for $s = -1$, there exists a finite number of basis states satisfying the condition $n_1 + n_2 + \cdots + n_r \leqslant k - 1$. The dimension is given, in this case, by $\frac{(k-1+r)!}{(k-1)!r!}$. This is exactly the dimension of the Fock representation of $A_r$ statistics discussed in [6]. This condition-restriction is closely related to the so-called generalized exclusion Pauli principle according to which no more than $k - 1$ particles can be accommodated in the same quantum state. In this sense, for $s = -1$, the generalized $A_r$ quantum statistics give statistics of fermionic behavior. They will be termed here as fermionic $A_r$ statistics and those corresponding to $s = 1$ will be named bosonic $A_r$ statistics. Having specified the Fock space associated with the generalized $A_r$ statistics, one can obtain the spectrum of Hamiltonian (6). For convenience, we set $c = \frac{2ks-s-1}{2r+2}$ in (7) and using the actions of creation and annihilation operators (9), 10), one has

$$H|n_1, \ldots, n_i, \ldots, n_r\rangle = \left(e_0 + \sum_{i=1}^{r} e_i n_i\right)|n_1, \ldots, n_i, \ldots, n_r\rangle. \tag{12}$$

It is remarkable that, for $s = -1$, the spectrum of $H$ is similar (with a slight modification) to energy eigenvalues of the $A_r$ Calogero model (see for instance equation (1.2) in [25]). The latter describes the dynamical model containing $r + 1$ particles on a line with long-range interactions and provides a microscopic realization of fractional statistics [13, 26]. It is important to stress that for $e_i = 0$ for all $i = 1, 2, \ldots, r$, the ground-state energy $e_0$ is degenerate (the degeneracy coincides with the dimension of the Fock space. It is finite (respectively infinite) for fermionic (respectively bosonic) $A_r$ statistics. It follows that $h$ in equation (6) can be considered as a potential responsible for the degeneracy lifting and inducing fluctuations around the ground-state energy $e_0$. This remark constitutes the key ingredient, as will be clarified later, to define $A_r$ quantum droplets and to derive their excitations. Finally, we point out that for large $k$, we have

$$[a_i^-, a_j^+] \approx k\delta_{ij} \tag{13}$$

reflecting that the generalized $A_r$ statistics (fermionic and bosonic ones) coincide with the Bose statistics and the Jacobson operators reduce to the Bose ones (creation and annihilation operators of harmonic oscillators).

Besides the Fock representation discussed in this section, it is interesting to look for analytical realizations of the space representation associated with the Fock representations of the generalized $A_r$ statistics. These realizations constitute a useful analytical tool in connection with variational and path integral methods to describe the quantum dynamics of the system described by the Hamiltonian $H$.

### 2.4. Bargmann realizations

First note that the Bargmann realization associated with $A_r$ statistics was derived in [18] (see also [19, 20]). Here, we recall some results needed for our task. This realization uses a suitably defined Hilbert space of entire analytical functions. The Jacobson annihilation generators $a_i^\pm$ are realized as first-order differential operators with respect to complex variables $z_i$

$$a_i^- \longrightarrow \frac{\partial}{\partial z_i}. \tag{14}$$

The key point of such analytical realization lies on the fact that we represent the Fock states $|n_1, n_2, \ldots, n_r\rangle$ as power of complex variables $z_1, z_2, \ldots, z_r$:

$$|n_1, n_2, \ldots, n_r\rangle \longrightarrow C_{n_1,\ldots,n_r} z_1^{n_1} z_2^{n_2} \cdots z_r^{n_r}. \tag{15}$$

Using the action of the annihilation operators on the Fock space $\mathcal{F}$ and the correspondences (14) and (15), the coefficients $C_{n_1,n_2,\ldots,n_r}$ are obtained as

$$C_{n_1,n_2,\ldots,n_r} = \left[\frac{(k-1+sn)!}{(k-1)!}\right]^{\frac{s}{2}} \frac{1}{\sqrt{n_1!}\sqrt{n_2!}\ldots\sqrt{n_r!}} \tag{16}$$

where $n = n_1 + n_2 + \cdots + n_r$ and $s = 1$ (respectively $s = -1$) for bosonic (respectively fermionic) $A_r$ statistics. Using equations (15), (16), one can determine the differential action of the Jacobson creation operators. Indeed, from the actions of the generators $a_i^+$ on the Fock space and the triple relations (3) and (4), one obtains

$$a_i^+ \longrightarrow \frac{1}{2}(2k+s-1)z_i + sz_i \sum_{j=1}^{r} z_j \frac{\mathrm{d}}{\mathrm{d}z_j}. \tag{17}$$

The Jacobson generators act as first-order linear differential operators. An arbitrary vector of the Fock space $\mathcal{F}$

$$|\phi\rangle = \sum_{n_1} \sum_{n_2} \cdots \sum_{n_r} \phi_{n_1,n_2,\ldots,n_r} |n_1, n_2, \ldots, n_r\rangle,$$

is realized as

$$\phi(z_1, z_2 \ldots, z_r) = \sum_{n_1} \sum_{n_2} \cdots \sum_{n_r} \phi_{n_1,n_2,\ldots,n_r} C_{n_1,n_2,\ldots,n_r} z_1^{n_1} z_2^{n_2} \cdots z_r^{n_r}. \tag{18}$$

The inner product of two functions $\phi$ and $\phi'$ is defined by

$$\langle\phi'|\phi\rangle = \iint \cdots \int \mathrm{d}^2z_1\, \mathrm{d}^2z_2 \cdots \mathrm{d}^2z_r\, \Sigma(k; z_1, z_2, \ldots, z_r) \phi'^{\star}(z_1, z_2, \ldots, z_r) \phi(z_1, z_2, \ldots, z_r). \tag{19}$$

The computation of the integration measure $\Sigma$, assumed to be isotropic, can be performed by choosing $|\phi\rangle = |n_1, n_2, \ldots, n_r\rangle$ and $|\phi'\rangle = |n_1', n_2', \ldots, n_r'\rangle$. A direct computation shows that the measure can be cast in the following compact form:

$$\Sigma(\varrho_1, \varrho_2, \ldots, \varrho_r) = \pi^{-r} \left[\frac{(k-1)!}{(k-sr+\frac{1}{2}(s-1))!}\right]^s \left[1 + s\left(\varrho_1^2 + \varrho_2^2 + \cdots + \varrho_r^2\right)\right]^{sk-r-\frac{1}{2}(s+1)} \tag{20}$$

where $\varrho_i = |z_i|^2$. One can write the function $\phi(z_1, z_2, \ldots, z_r)$ as the product of the state $|\phi\rangle$ with some ket $|\bar{z}_1, \bar{z}_2 \ldots, \bar{z}_r\rangle$ labeled by the complex conjugate of the variables $z_1, z_2, \ldots, z_r$

$$\phi(z_1, z_2, \ldots, z_r) = \mathcal{N}\langle\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_r|\phi\rangle, \tag{21}$$

where $\mathcal{N}$ is a normalization constant to be adjusted later. Taking $|\phi\rangle = |n_1, n_2, \ldots, n_r\rangle$, we have

$$\langle\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_r|n_1, n_2, \ldots, n_r\rangle = \mathcal{N}^{-1} C_{n_1,n_2,\ldots,n_r} z_1^{n_1} z_2^{n_2} \cdots z_r^{n_r}. \tag{22}$$

This implies

$$|z_1, z_2, \ldots, z_r\rangle = \mathcal{N}^{-1} \sum_{n_1} \sum_{n_2} \cdots \sum_{n_r} \left[\frac{(k-1+sn)!}{(k-1)!}\right]^{\frac{s}{2}} \frac{z_1^{n_1}}{\sqrt{n_1!}} \frac{z_2^{n_2}}{\sqrt{n_2!}} \cdots \frac{z_r^{n_r}}{\sqrt{n_r!}}. \tag{23}$$

It is important to note that expansion (23) converges for bosonic $A_r$ statistics when $\bar{z}.z = |z_1|^2 + |z_2|^2 + \cdots + |z_r|^2 < 1$. In other words, the complex variables $z_1, z_2, \ldots, z_r$ should be in the complex domain defined by $\{(z_1, z_2, \ldots, z_r) : |z_1|^2 + |z_2|^2 + \cdots + |z_r|^2 < 1\}$. The normalization constant in (23) is given by

$$\mathcal{N} = (1 - s(|z_1|^2 + |z_2|^2 + \cdots + |z_r|^2))^{-\frac{1}{4}(2ks-s+1)}. \tag{24}$$

States (23) are continuous in the labeling, constitute an over complete set in respect of the measure given by (20) and then are coherent in the Klauder–Perelomov sense.

## 3. Semi-classical analysis

One of the uses of the above Bargmann realizations is that they provide us with a simple way to establish a correspondence between operators and classical functions on the phase space of the systems under consideration. So, in this section we shall investigate the semi-classical properties of $A_r$ statistical systems (bosonic as well as fermionic) in the Bargmann space for $k$ large.

### 3.1. The density matrix and Husimi distribution

It is commonly accepted that the exploration of the classical behavior of any quantum system hinges on whether one can describe the behavior of the wavefunctions in terms of a density matrix. So, let $N = N_1 + N_2 + \cdots + N_k$ be the number of quantas of the system where $N_i$ stands for the particle number in the mode $i$. The corresponding density operator is

$$\rho_0 = \sum_{n_1}^{N_1} \sum_{n_2}^{N_2} \cdots \sum_{n_r}^{N_r} |n_1, n_2, \ldots, n_r\rangle \langle n_1, n_2, \ldots, n_r|. \tag{25}$$

In the Bargmann space, the mean value of the density matrix is defined by

$$\rho_0(\bar{z}, z) = \langle z|\rho_0|z \rangle, \tag{26}$$

where $z$ stands for the variables $(z_1, z_2, \ldots, z_r)$ labeling the coherent states for $A_r$ statistics systems. The mean value $\rho_0(\bar{z}, z)$ is the symbol associated with the density operator and it can be identified with the Husimi distribution for $A_r$ quantum systems. As we are concerned with the situation when $k$ is large, let us investigate the spacial shape of the mean value of the density operator. For $A_r$ bosonic statistics, we have

$$\rho_0(\bar{z}, z) = (1 - \bar{z}.z)^k \sum_{n_1=0}^{N_1} \cdots \sum_{n_r=0}^{N_r} \frac{(k-1+n_1\cdots+n_k)!}{(k-1)!n_1!\cdots n_r!} |z_1|^{2n_1} \cdots |z_k|^{2n_r}. \tag{27}$$

It is easy to see that, for $k$ large, the identity

$$(1 - \bar{z}.z)^{-k} = \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \frac{(k-1+n_1\cdots+n_k)!}{(k-1)!n_1!\cdots n_r!} |z_1|^{2n_1} \cdots |z_k|^{2n_r}, \tag{28}$$

gives

$$(1 - \bar{z}.z)^k = \exp(-k\bar{z}.z). \tag{29}$$

Furthermore, using the relation

$$\sum_{n_1=0}^{N_1} \cdots \sum_{n_r=0}^{N_r} \frac{(k-1+n_1\cdots+n_r)!}{(k-1)!n_1!\cdots n_r!} |z_1|^{2n_1} \cdots |z_r|^{2n_r} = \sum_{n=0}^{N} \frac{(k-1+n)!}{(k-1)!(n)!} (\bar{z}.z)^n \tag{30}$$

where $n = n_1 + \cdots + n_r$, one can see the term involving the sum in the expression of $\rho_0$ behaves like

$$\sum_{n=0}^{N} \frac{(k\bar{z}.z)^n}{n!}. \tag{31}$$

It follows that, for $k$ large, the density can be approximated by

$$\rho_0(\bar{z}, z) \simeq \exp(-k\bar{z}.z) \sum_{n=0}^{N} \frac{(k\bar{z}.z)^n}{n!} \simeq \Theta(N - k\bar{z}.z) \tag{32}$$

for a large number $N$ of particles. Clearly, $\rho_0(\bar{z}, z)$ is a step function for $k \longrightarrow \infty$ and $N \longrightarrow \infty$ ($\frac{N}{k}$ fixed). Similarly, the classical density for $A_r$ fermionic statistics

$$\rho_0(\bar{z}, z) = (1 + \bar{z}.z)^{-\frac{k-1}{2}} \sum_{n_1=0}^{N_1} \cdots \sum_{n_r=0}^{N_r} \frac{(k-1)}{(k-1-n))!n_1! \cdots n_r!} |z_1|^{2n_1} \cdots |z_r|^{2n_r} \tag{33}$$

gives for large $k$ and $N$

$$\rho_0(\bar{z}, z) \simeq \Theta(N - k\bar{z}.z). \tag{34}$$

It corresponds to a droplet configuration with the boundary defined by $k\bar{z}.z = N$ and its radius proportional to $\sqrt{N}$. The derivative of this density tends to a $\delta$ function. As we will see an interesting outcome of the semi-quantal dynamics happens when the parameter $k$ tends to infinity. This indicates that $k$ plays a crucial role in determining the classical limit of $A_r$ quantum dynamics and in deriving the edge excitations for $A_r$ fermionic as well as bosonic statistics.

### 3.2. The star product and Moyal bracket

A second necessary ingredient to perform our semi-classical analysis is the star product. In fact, as we will discuss next, for $k$ large the mean value of the product of two operators leads to the Moyal star product. To show this, for every operator $A$ acting on the Fock space $\mathcal{F}$, we associate the function

$$\mathcal{A}(\bar{z}, z) = \langle z|A|z \rangle. \tag{35}$$

An associative star product of two functions $\mathcal{A}(\bar{z}, z)$ and $\mathcal{B}(\bar{z}, z)$ is defined by

$$\mathcal{A}(\bar{z}, z) \star \mathcal{B}(\bar{z}, z) = \langle z|AB|z \rangle = \int d\mu(\bar{z}', z') \langle z|A|z' \rangle \langle z'|B|z \rangle \tag{36}$$

where the measure $d\mu(\bar{z}, z) = d^2z_1 \, d^2z_2 \cdots d^2z_r \, \Sigma$ is given by equation (20). To compute this star product, let us exploit the analytical properties of coherent states defined above. Indeed, using equations (23) and (24), one can see that the function defined by

$$\mathcal{A}(\bar{z}', z) = \frac{\langle z'|A|z \rangle}{\langle z'|z \rangle} \tag{37}$$

satisfies the following holomorphic and anti-holomorphic conditions:

$$\frac{\partial}{\partial \bar{z}_i} \mathcal{A}(\bar{z}', z) = 0 \qquad \frac{\partial}{\partial z_i'} \mathcal{A}(\bar{z}', z) = 0 \tag{38}$$

for $i = 1, 2, \ldots, r$ and $z \neq z'$. Consequently, the action of the translation operator on the function $\mathcal{A}(\bar{z}', z)$ gives

$$\exp\left(z'.\frac{\partial}{\partial z}\right) \mathcal{A}(\bar{z}', z) = \mathcal{A}(\bar{z}', z + z') \tag{39}$$

from which one can see that the function $\mathcal{A}(\bar{z}, z')$ is given by

$$\exp\left(-z.\frac{\partial}{\partial z'}\right) \exp\left(z'.\frac{\partial}{\partial z}\right) \mathcal{A}(\bar{z}, z) = \exp\left((z'-z).\frac{\partial}{\partial z}\right) \mathcal{A}(\bar{z}, z) = \mathcal{A}(\bar{z}, z') \tag{40}$$

in terms of the function $\mathcal{A}(\bar{z}, z)$. Similarly, one obtains

$$\exp\left(-\bar{z}.\frac{\partial}{\partial \bar{z}'}\right) \exp\left(\bar{z}'.\frac{\partial}{\partial \bar{z}}\right) \mathcal{A}(\bar{z}, z) = \mathcal{A}(\bar{z}', z). \tag{41}$$

Equivalently, equations (40) and (41) can also be cast in the following forms:

$$\exp\left((z'-z).\frac{\partial}{\partial z}\right)\mathcal{A}(\bar{z},z) = \mathcal{A}(\bar{z},z') \tag{42}$$

and

$$\exp\left((\bar{z}'-\bar{z}).\frac{\partial}{\partial \bar{z}}\right)\mathcal{A}(\bar{z},z) = \mathcal{A}(\bar{z}',z) \tag{43}$$

respectively. Combining equations (36), (37) and (42), (43), the star product rewrites as

$$\mathcal{A}(\bar{z},z) \star \mathcal{B}(\bar{z},z) = \int d\mu(\bar{z}',z') \exp\left((z'-z).\frac{\partial}{\partial z}\right)\mathcal{A}(\bar{z},z)|\langle z|z'\rangle|^2 \exp\left((\bar{z}'-\bar{z}).\frac{\partial}{\partial \bar{z}}\right)\mathcal{B}(\bar{z},z) \tag{44}$$

where the overlapping of coherent states is given by

$$\langle z|z'\rangle = [(1-s\bar{z}'.z')(1-s\bar{z}.z)(1-s\bar{z}'.z)^{-2}]^{\frac{2ks+1-s}{4}} \tag{45}$$

with $s = +1, -1$ corresponding to bosonic and fermionic statistics, respectively. Clearly, the modulus of kernel (45) possesses the properties $|\langle z|z'\rangle| = 1$ if and only if $z = z'$, $|\langle z|z'\rangle| < 1$ and $|\langle z|z'\rangle| \to 0$ for $k$ large. The latter properties are helpful to get the star product between two functions on the Bargmann space. In this respect, we introduce a function $s(z',z)$ of the coordinates of two points on the Bargmann space

$$s^2(z',z) = -\ln|\langle z|z'\rangle|^2 = \frac{1}{2}(2ks-s+1)\ln\frac{(1-s\bar{z}'.z)(1-s\bar{z}.z')}{(1-s\bar{z}.z)(1-s\bar{z}'.z')}. \tag{46}$$

It verifies the properties : $s(z',z) = s(z,z')$ and $s(z',z) = 0$ if and only if $z' = z$. This function can be interpreted as the distance between two points on the Bargmann space. It turns out that overlapping (45) generates the metric. In fact the line element $ds^2$, defined as the quadratic part of the decomposition of $s^2(z,z+dz)$ (distance between two infinitesimal points), is given by

$$ds^2 = g_{i\bar{j}}\,dz_i\,d\bar{z}_j \tag{47}$$

where summation over repeated indices is understood and the components of the metric $g_{i\bar{j}}$ are defined as

$$g_{i\bar{j}} = \left(k+\frac{s}{2}-\frac{1}{2}\right)\left[\frac{\delta_{ij}}{1-s\bar{z}.z}+s\frac{\bar{z}_i z_j}{(1-s\bar{z}.z)^2}\right]. \tag{48}$$

We now come to the evaluation of the star product for $k$ large. Since $s^2(z',z)$ tends to infinity with $k \to \infty$ if $z \neq z'$ and equals zero if $z = z'$, one can conclude that, in that limit, the domain $z \simeq z'$ gives only a contribution to integral (44). Decomposing the integrand near the point $z \simeq z'$ and going to integration over $\eta = z'-z$, one gets

$$\mathcal{A}(\bar{z},z) \star \mathcal{B}(\bar{z},z) = \int\frac{d\eta.d\bar{\eta}}{\pi^r}\exp\left(\eta.\frac{\partial}{\partial z}\right)\mathcal{A}(\bar{z},z)\exp\left(-g_{i\bar{j}}\eta_i\bar{\eta}_{\bar{j}}\right)\exp\left(\bar{\eta}.\frac{\partial}{\partial \bar{z}}\right)\mathcal{B}(\bar{z},z). \tag{49}$$

It follows that the star product between two functions on the Bargmann space associated with $A_r$ statistics is given by

$$\mathcal{A}(\bar{z},z) \star \mathcal{B}(\bar{z},z) = \mathcal{A}(\bar{z},z)\mathcal{B}(\bar{z},z) - g^{i\bar{j}}\frac{\partial\mathcal{A}}{\partial z_i}(\bar{z},z)\frac{\partial\mathcal{B}}{\partial \bar{z}_j}(\bar{z},z) + O\left(\frac{1}{k^2}\right) \tag{50}$$

where the matrix

$$g^{i\bar{j}} = 2\frac{1-s\bar{z}.z}{2k+s-1}(\delta_{ij}-sz_i\bar{z}_j) \tag{51}$$

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is the inverse of the metric $g_{i\bar{j}}$ and is proportional to $1/k$. Then, the symbol or function associated with the commutator of two operators $A$ and $B$ is given by

$$\langle z|[A, B]|z\rangle = \{\mathcal{A}(\bar{z}, z), \mathcal{B}(\bar{z}, z)\}_\star = -g^{i\bar{j}}\left(\frac{\partial \mathcal{A}}{\partial z_i}(\bar{z}, z)\frac{\partial \mathcal{B}}{\partial \bar{z}_j}(\bar{z}, z) - \frac{\partial \mathcal{B}}{\partial z_i}(\bar{z}, z)\frac{\partial \mathcal{A}}{\partial \bar{z}_j}(\bar{z}, z)\right) \quad (52)$$

where

$$\{\mathcal{A}(\bar{z}, z), \mathcal{B}(\bar{z}, z)\}_\star = \mathcal{A}(\bar{z}, z) \star \mathcal{B}(\bar{z}, z) - \mathcal{B}(\bar{z}, z) \star \mathcal{A}(\bar{z}, z). \quad (53)$$

is the so-called the Moyal bracket.

### 3.3. The excitation potential

The quantum droplet under consideration is specified by the density matrix $\rho_0$ (25). The excitations of this configuration can be described by a unitary time evolution operator $U$ which gives information concerning the dynamics of the excitations around $\rho_0$. The excited states will be characterized by a density operator $\rho = U\rho_0 U^\dagger$. In this respect, the Hamiltonian $h$ in equation (6) may be viewed as the excitation potential of the quantum droplet. Indeed, as we mentioned in the previous section, in the absence of $h$ (all $e_i$ vanishing) the states $|n_1, n_2, \ldots, n_r\rangle$ are eigenstates of $H$ with the same eigenvalue $e_0$. The degeneracy of the energy $e_0$ coincides with the dimension of the Fock space $\mathcal{F}$. It is finite for $A_r$ bosonic systems and takes a finite value for $A_r$ fermionic statistics. The Hamiltonian $h$ is exactly the excitation potential that induces a degeneracy lifting. Using expressions (12) and (23), the mean value of the excitation potential $h$ is

$$\langle z|h|z\rangle = \mathcal{H}(\bar{z}, z) = \left(k + \frac{s}{2} - \frac{1}{2}\right)\sum_{i=1}^{r} e_i \frac{z_i \bar{z}_i}{1 - s\bar{z}.z}. \quad (54)$$

The function $\mathcal{H}(\bar{z}, z)$ is the symbol associated with $h$. As next we are concerned by the edge excitations living on the boundary of the $A_r$ quantum droplet, it is simply verified from equations (23) and (24) that the Hamiltonian symbol $\mathcal{H}$ takes for $k$ large the simple form

$$\mathcal{H}(\bar{z}, z) = k\sum_{i=1}^{r} e_i z_i \bar{z}_i, \quad (55)$$

which is just the classical harmonic oscillator potential.

## 4. Edge excitations and chiral bosons action

### 4.1. Effective action

In this section, we derive the effective action for excitations living on the edge of an $A_r$ quantum droplet. The derivation is based on semi-classical analysis given in the previous section. As mentioned above, the dynamical information, related to degrees of freedom of the edge states, is contained in the unitary operator $U$. The corresponding action is

$$S = \int dt \, \text{Tr}(\rho_0 U^\dagger(i\partial_t - H)U). \quad (56)$$

It is compatible with the Liouville evolution equation for the density matrix

$$i\frac{\partial \rho}{\partial t} = [H, \rho].$$

To write an effective action describing the edge excitations, we evaluate the quantities occurring in (56) as classical functions on the basis of the semi-classical analysis performed above. We

start by computing the term $\mathrm{i} \int \mathrm{d}t \, \mathrm{Tr}(\rho_0 U^\dagger \partial_t U)$. For this, we set $U = \mathrm{e}^{+\mathrm{i}\Phi}$ $(\Phi^\dagger = \Phi)$. A direct computation gives

$$\mathrm{d}U = \sum_{n=1}^{\infty} \frac{(\mathrm{i})^n}{n!} \sum_{p=0}^{n-1} \Phi^p \, \mathrm{d}\Phi \, \Phi^{n-1-p}, \tag{57}$$

from which one obtains

$$U^\dagger \, \mathrm{d}U = \mathrm{i} \int_0^1 \mathrm{d}\alpha \, \mathrm{e}^{-\mathrm{i}\alpha\Phi} \, \mathrm{d}\Phi \, \mathrm{e}^{+\mathrm{i}\alpha\Phi}. \tag{58}$$

Thus, we have

$$\mathrm{e}^{-\mathrm{i}\Phi} \partial_t \, \mathrm{e}^{+\mathrm{i}\Phi} = \mathrm{i} \int_0^1 \mathrm{d}\alpha \, \mathrm{e}^{-\mathrm{i}\alpha\Phi} \partial_t \Phi \, \mathrm{e}^{+\mathrm{i}\alpha\Phi}. \tag{59}$$

Using the Baker–Campbell–Hausdorff formula, one can show

$$\mathrm{i} \int \mathrm{d}t \, \mathrm{Tr}(\rho_0 U^\dagger \partial_t U) = \int \mathrm{d}t \sum_{n=0}^{\infty} \frac{-(\mathrm{i})^n}{(n+1)!} \, \mathrm{Tr}\left(\underbrace{[\Phi, \ldots [\Phi}_{n}, \rho_0] \cdots] \partial_t \Phi\right). \tag{60}$$

Due to the coherent states completeness, the trace of any operator $A$ is

$$\mathrm{Tr}\, A = \int \mathrm{d}\mu(\bar{z}, z) \langle z|A|z\rangle.$$

It follows that equation (60) rewrites as

$$\mathrm{i} \int \mathrm{d}t \, \mathrm{Tr}(\rho_0 U^\dagger \partial_t U) = \int \mathrm{d}\mu \, \mathrm{d}t \sum_{n=0}^{\infty} \frac{-(\mathrm{i})^n}{(n+1)!} \, \underbrace{\{\Phi, \ldots \{\Phi}_{n}, \rho_0\}_\star \cdots\}_\star \star \partial_t \Phi \tag{61}$$

where the star product and the Moyal bracket are respectively defined by (50) and (52). It is important to stress that $\rho_0$ and $\Phi$ in equation (61) are now classical functions. It is easy to see that equation (61) gives

$$\mathrm{i} \int \mathrm{d}t \, \mathrm{Tr}(\rho_0 U^\dagger \partial_t U) \simeq -\frac{\mathrm{i}}{2} \int \mathrm{d}\mu \, \mathrm{d}t \{\Phi, \rho_0\}_\star \partial_t \Phi \tag{62}$$

where we have dropped terms in $\frac{1}{k^2}$ as well as the total time derivative. Using expression (52), the Moyal bracket in (62) writes

$$\{\Phi, \rho_0\}_\star = \frac{2\mathrm{i}}{2k+s-1}(\mathcal{L}\Phi)\frac{\partial \rho_0}{\partial(\bar{z}.z)} \tag{63}$$

where the first-order differential operator is

$$\mathcal{L} = \mathrm{i}(1 - s\bar{z}.z)^2 \left(z.\frac{\partial}{\partial z} - \bar{z}.\frac{\partial}{\partial \bar{z}}\right). \tag{64}$$

For $k$ large the density function (see equations (32) and (34)) is a step function. Its derivative is a $\delta$ function with support on the boundary $\partial\mathcal{D}$ of the droplet $\mathcal{D}$ defined by $k\bar{z}.z = N$. It follows:

$$\mathrm{i} \int \mathrm{d}t \, \mathrm{Tr}(\rho_0 U^\dagger \partial_t U) \approx -\frac{1}{2} \int \mathrm{d}\mu \, \mathrm{d}t \, \delta(N - k\bar{z}.z)(\mathcal{L}\Phi)(\partial_t \Phi) = -\frac{1}{2} \int_{\partial\mathcal{D} \times \mathbf{R}^+} \mathrm{d}t \, (\mathcal{L}\Phi)(\partial_t \Phi). \tag{65}$$

The second step in the derivation of edge states action consists in the simplification of the second term in (56) involving $H$. By a straightforward calculation, we obtain

$$\mathrm{Tr}(\rho_0 U^\dagger h U) = \mathrm{Tr}(\rho_0 h) + \mathrm{i} \, \mathrm{Tr}([\rho_0, h]\Phi) + \tfrac{1}{2} \, \mathrm{Tr}([\rho_0, \Phi][h, \Phi]). \tag{66}$$

The first term on rhs of (66) is $\Phi$-independent. We drop it since it does not contain any information about the dynamics of the edge excitations. The second term on the rhs of (66) rewrites as

$$\mathrm{i}\,\mathrm{Tr}([\rho_0, h]\Phi) \approx \mathrm{i} \int \mathrm{d}\mu \{\rho_0, \mathcal{H}\}_\star \Phi \tag{67}$$

in terms of the Moyal bracket where $\mathcal{H}$ is given by (55). By a direct computation, one can see that

$$\mathrm{i}\,\mathrm{Tr}([\rho_0, h]\Phi) \longrightarrow 0. \tag{68}$$

The last term on the rhs of (66) gives

$$\frac{1}{2}\,\mathrm{Tr}([\rho_0, \Phi][h, \Phi]) \approx \frac{\mathrm{i}}{2k} \int \mathrm{d}\mu \, \mathrm{d}t \, (\mathcal{L}\Phi) \frac{\partial\rho_0}{\partial(\bar{z}.z)} \{\mathcal{H}, \Phi\}_\star \tag{69}$$

where the Moyal bracket is given by

$$\{\mathcal{H}, \Phi\}_\star = \frac{2k(1 - s\bar{z}.z)}{2k + s - 1} \left[ s \left( z.\frac{\partial\Phi}{\partial z} - \bar{z}.\frac{\partial\Phi}{\partial\bar{z}} \right) \sum_{i=1}^{r} e_i \bar{z}_i z_i - \sum_{i=1}^{r} e_i \left( z_i \frac{\partial\Phi}{\partial z_i} - \bar{z}_i \frac{\partial\Phi}{\partial\bar{z}_i} \right) \right]. \tag{70}$$

Since the derivative of the density $\rho_0$ gives a delta function with support on the boundary of the quantum droplet, equation (69) is simplified

$$\{\mathcal{H}, \Phi\}_\star \approx \mathrm{i} \sum_{i=1}^{r} e_i \mathcal{L}_i \Phi \tag{71}$$

where $\mathcal{L}_i$ is the angular momentum with respect to the variable $z_i$ (see definition (64)). Finally, we obtain

$$\int \mathrm{d}t \, \mathrm{Tr}(\rho_0 U^\dagger H U) = \frac{1}{2} \int \mathrm{d}\mu \, \mathrm{d}t \, \delta(N - k\bar{z}.z)(\mathcal{L}\Phi) \sum_{i=1}^{r} e_i(\mathcal{L}_i\Phi) + O\left(\frac{1}{k^2}\right). \tag{72}$$

Note that we have eliminated the term containing the ground-state energy $e_0$ which does not contribute to the edge dynamics. Combining (65) and (72), we get

$$S \approx -\frac{1}{2} \int_{\partial\mathcal{D}\times\mathbf{R}^+} \mathrm{d}\mu \, \mathrm{d}t \, \delta(N - k\bar{z}.z)(\mathcal{L}\Phi) \left( (\partial_t\Phi) + \sum_{i=1}^{r} e_i(\mathcal{L}_i\Phi) \right). \tag{73}$$

This action involves only the time derivative of $\Phi$ and the tangential derivatives $(\mathcal{L}_i\Phi)$. It is a generalization of a chiral abelian Wess–Zumino–Witten (WZW) theory [22]. It is interesting to note that for $r = 1$, we recover the WZW action describing a bosonized theory of a system of large fermions in two dimension [24]. Solving the equations of motion arising from action (73) gives the nature of edge states. This will be done in the following subsection.

### 4.2. Edge fields

Action (73) is minimized by the fields $\Phi$ that satisfy the equation of motion

$$\mathcal{L} \left( \partial_t\Phi + \sum_{i=1}^{r} e_i \mathcal{L}_i \Phi \right) = 0. \tag{74}$$

Since the theory is defined on the boundary of the droplet fixed by the condition $\bar{z}.z = \frac{N}{k}$, we introduce the angular variables $\theta_i \left( z_i = \sqrt{\frac{N}{k}} \mathrm{e}^{\mathrm{i}\theta_i} \right)$. This is the most simple parametrization

that one can consider. The operators $\mathcal{L}_i$ reduce to partial derivatives $\partial_i$ with respect to $\theta_i$. The general solution of the equation of motion can be written as

$$\Phi(\theta_1, \theta_2, \ldots, \theta_r, t) = \Phi(\theta_1 - e_1 t, \theta_2 - e_2 t, \ldots, \theta_r - e_r t) + \Lambda(t). \tag{75}$$

In the last equation, $\Lambda(t)$ represents the gauge degree of freedom corresponding to the invariance of action (73) under the transformation

$$\Phi \longrightarrow \Phi + \lambda(t).$$

This can be discarded by imposing the gauge condition

$$\left( \partial_t + \sum_{i=1}^{r} e_i \partial_i \right) \Phi = 0. \tag{76}$$

Next, we assume that the field $\Phi$ can be expressed in a factorized form

$$\Phi = \Phi_1 \Phi_2 \cdots \Phi_r \tag{77}$$

in terms of $r$ components $\Phi_i = \Phi_i(\theta_i, t)$ $(i = 1, 2, \ldots, r)$ satisfying the equations

$$(\partial_t + e_i \partial_i)\Phi_i = 0. \tag{78}$$

It is remarkable that equations (77) and (78) are compatible with the gauge-fixing condition (76). To obtain the solution of (78), we assume that the field $\Phi_i$ satisfies the following periodicity condition:

$$\Phi_i(2\pi, t) - \Phi_i(0, t) = -2\pi \alpha_0^i \tag{79}$$

where $\alpha_0^i$ is a time-independent constant. It is easy to see that the general solution of (78) is then given by

$$\Phi_i(\theta_i, t) = \bar{\alpha}_0^i - \alpha_0^i(\theta_i - e_i t) + \mathrm{i} \sum_{n \neq 0} \frac{\alpha_n^i}{n} \, \mathrm{e}^{\mathrm{i}n(\theta_i - e_i t)} \tag{80}$$

where the constant $\bar{\alpha}_0^i$ can be viewed as the canonical momentum associated with $\alpha_0^i$. Note also that the complex coefficients in (80) satisfy $(\alpha_n^i)^\star = \alpha_{-n}^i$ required by the reality condition of the field $\Phi_i$. The canonical momentum corresponding to the field $\Phi_i$ is

$$\Pi_i(\theta_i, t) = \alpha_0^i + \sum_{n \neq 0} \alpha_n^i \, \mathrm{e}^{\mathrm{i}n(\theta_i - e_i t)}. \tag{81}$$

The quantization of the theory of edge excitations described by action (73) can be performed by imposing the equal time commutation rules

$$[\Pi_i(\theta_i, t), \Phi_j(\theta_j, t)] = \mathrm{i}\delta_{ij}\delta(\theta_i - \theta_j). \tag{82}$$

This implies that $\bar{\alpha}_0^i$, $\alpha_0^i$ and $\alpha_n^i$ become operators satisfying the relations

$$\left[\alpha_n^i, \alpha_m^j\right] = \delta_{ij}\delta_{m+n,0} \qquad \left[\alpha_0^i, \bar{\alpha}_0^j\right] = i\delta_{ij}. \tag{83}$$

The other commutators vanish. This reflects that each field $\Phi_i$ is a superposition of oscillating modes on the boundary of the $A_r$ droplet. For a fixed $i$, the Hilbert space $\mathcal{H}_i$ is a tensorial product of harmonic oscillator Fock spaces. The whole Hilbert space is then given by

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \cdots \otimes \mathcal{H}_r. \tag{84}$$

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## 5. Concluding remarks

To conclude, let us summarize the main points discussed in this paper. We provided a general approach of what we agreed to call 'chiral boson theory on Bargmann space associated with $A_r$ quantum statistics'. We started discussing the essential structures of these new quantum statistics. We also gave the analytical Bargmann realizations. In the Bargmann space, the problem of computing commutators has been rephrased in terms of (more easy) Moyal brackets of functions associated with the algebra generated by creation and annihilation (Jacobson) operators. The moyal bracket captures the essence of the full quantum characteristics of $A_r$ statistics. This potentially provides us with an important tool to obtain semi-classically the effective action describing a large collection of $N$ particles with $A_r$ statistics. More precisely, we have shown that for large $N$ and large $k$ ($k$ the parameter indexing the Fock representations), the system behaves like a droplet in the Bargmann space (phase space). We derived the effective action (equation (73)) describing the excitations living on the droplet's boundary. It is remarkable that the obtained action is similar to the Wess–Zumino action [22]. As a by product, we have shown that the boundary excitations are essentially a tensorial product of $r$ bosonic fields (cf equation (77)). Each bosonic field is given in terms of infinite oscillating modes (harmonic oscillators) (see equation (80)).

The results of the present paper can be used in relation with many-body Wigner quantum systems [27]. They can also be used in connection with the so-called generalized spin systems [28]. In fact, the usual $SU(2)$ spin systems are extended to spin models based on an arbitrary Lie group. In particular, for the classical Lie algebras of class $A_r$, the $SU(2)$ spin generators are replaced by the $A_r$ generators which coincide with the creation and annihilation (Jacobson) operators in the terminology of this paper. It follows that the generalized spin models, discussed in [28], can be viewed as example of systems with generalized $A_r$ statistics. In this sense, we believe that the approach developed here can be adapted to the generalized spin systems. We hope to report on these issues in a forthcoming work.

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